VI. DESCRIPTIONS OF POLARIZED LIGHT[22]

Consider a **totally coherent** wave propagating in the positive direction

$$\mathbf{E}_{\mathbf{x}} = \mathbf{E}_x^0 \cos(\omega t + \delta_x) = \mathbf{E}_x^0 \left[\exp\left[i(\omega t + \delta_x)\right] + c.c.\right] \qquad \text{[VI-1a]}$$

$$\mathbf{E}_y = \mathbf{E}_y^0 \cos(\omega t + \delta_y) = \mathbf{E}_y^0 \left[\exp\left[i(\omega t + \delta_y)\right] + c.c.\right] \qquad \text{[VI-1b]}$$

For later reference, we note that the *full* (non-normalized) Jones vector representation[23] of this field is given by

$$\vec{\mathbf{J}} = \begin{bmatrix} \mathbf{E}_x^0 \exp\, i\delta_x \\ \mathbf{E}_y^0 \exp\, i\delta_y \end{bmatrix} \qquad \text{[VI-2]}$$

We can be easily shown that

$$\cos\omega t = \left[\sin(\delta_y - \delta_x)\right]^{-1} \left[\frac{\mathbf{E}_{\mathbf{x}}}{\mathbf{E}_x^0} \sin\delta_y - \frac{\mathbf{E}_y}{\mathbf{E}_y^0} \sin\delta_x\right] \qquad \text{[VI-3a]}$$

$$\sin\omega t = \left[\sin(\delta_y - \delta_x)\right]^{-1} \left[\frac{\mathbf{E}_{\mathbf{x}}}{\mathbf{E}_x^0} \cos\delta_y - \frac{\mathbf{E}_y}{\mathbf{E}_y^0} \cos\delta_x\right] \qquad \text{[VI-3b]}$$

---

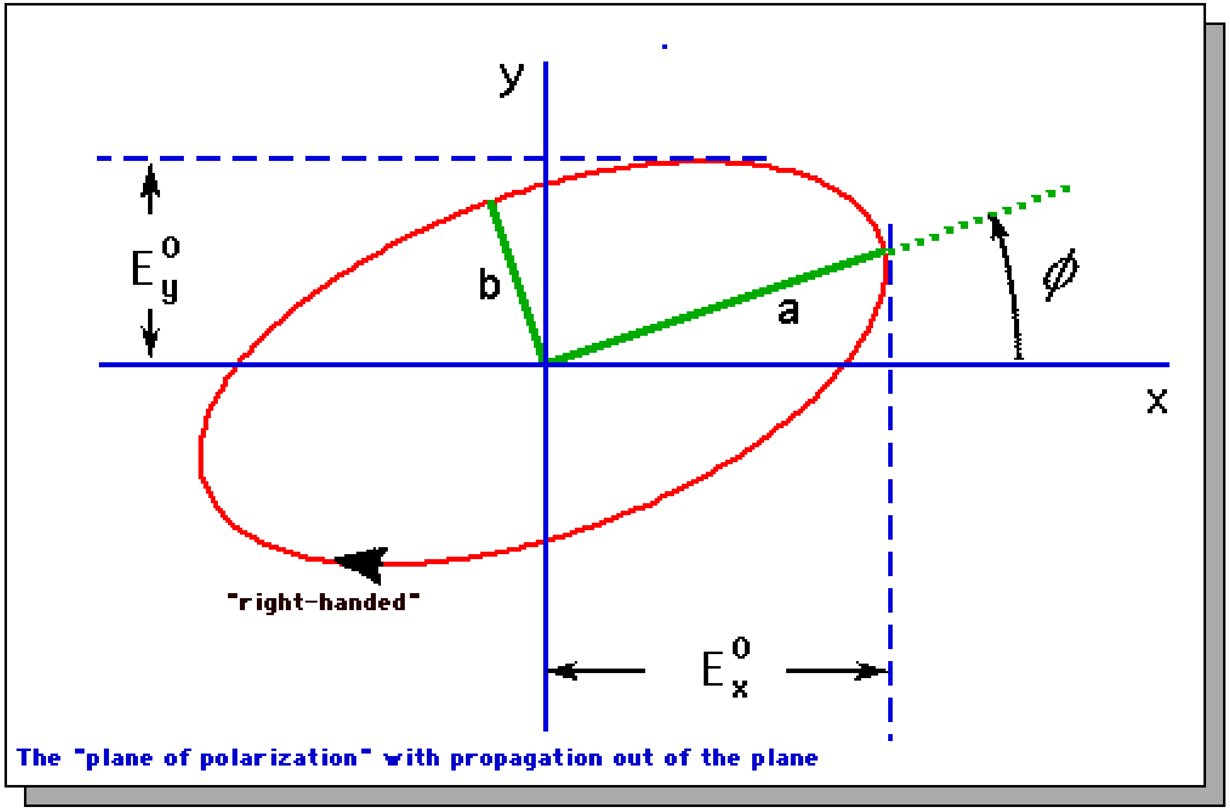
22 The best references on this subject are the following: 1.) William A. Shurcliff, *Polarized Light: Production and Use*, Harvard University Press (1962); 2.) D. Clarke and J.F. Grainger, *Polarized Light and Optical Measurement*, Pergamon Press (1971); 3.) Max Born and Emil Wolf, *Principles of Optics*, Pergamon Press (Particularly Section 1.4).

23 R. Clarke Jones, "New calculus for the treatment of optical systems. I. Description and discussion of calculus," *J. Opt. Soc. Amer. 31*, 488 (1941). To quote Shurcliff,

> "The Jones vector, … describes a polarized beam with the maximum algebraic brevity, and is eminently suited to the solving of problems involving beams whose phase relations phase relations are important. …"

Thus the locus of time sequence of fields in a plane perpendicular to the direction of propagtion follows an ellipse -- *viz*

$$\left(\frac{\mathbf{E}_x}{\mathbf{E}_x^0}\right)^2+\left(\frac{\mathbf{E}_y}{\mathbf{E}_y^0}\right)^2-2\left(\frac{\mathbf{E}_x}{\mathbf{E}_x^0}\right)\left(\frac{\mathbf{E}_y}{\mathbf{E}_y^0}\right)\cos\left(\delta_y-\delta_x\right)=\sin^2\left(\delta_y-\delta_x\right) \qquad \text{[VI-4]}$$



The "plane of polarization" with propagation out of the plane

For an electric field vector "seen" to be rotating in a clockwise direction by an observer receiving the radiation (*i.e.*, $\pi > \left(\delta_y-\delta_x\right) > 0$), the polarization is said to be *right-handed*. For rotation in the anticlockwise sense (*i.e.*, $0 > \left(\delta_y-\delta_x\right) > -\pi$), the polarization is said to be *right-handed*. Although this is a complete decription of the coherent field, it is not a

convenient one. It is useful to transform this equation to its principal axes form by the following transformation

$$\mathbf{E}_{\mathbf{x}} = \mathbf{E}_{\xi} \cos\varphi - \mathbf{E}_{\zeta} \sin\varphi \qquad \text{[VI-5a]}$$

$$\mathbf{E}_{y} = \mathbf{E}_{\zeta} \cos\varphi + \mathbf{E}_{\xi} \sin\varphi \qquad \text{[VI-5b]}$$

where we choose φ so that

$$\left(\frac{\mathbf{E}_{\xi}}{a}\right)^2 + \left(\frac{\mathbf{E}_{\zeta}}{b}\right)^2 = 1 \qquad \text{[VI-6a]}$$

$$\mathbf{E}_{\xi} = a\cos\left(\omega t + \delta_0\right) \qquad \text{[VI-6b]}$$

$$\mathbf{E}_{\zeta} = \pm\, b\sin\left(\omega t + \delta_0\right) \qquad \text{[VI-6c]}$$

Where the upper and lower signs are for, respectively, *right-handed* and *left-handed* polarizations. After quite a bit of algebraic manipulation, we find a more elegant and convenient description of polarization in terms of the following set of relationships

$$\left(\mathbf{E}_x^0\right)^2 + \left(\mathbf{E}_y^0\right)^2 = a^2 + b^2 \qquad \text{[VI-7a]}$$

$$\pm\, a\, b = \left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\sin\left(\delta_y - \delta_x\right) = \left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\sin\delta \qquad \text{[VI-7b]}$$

$$\tan 2\varphi = \frac{2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\cos\left(\delta_y - \delta_x\right)}{\left(\mathbf{E}_x^0\right)^2 - \left(\mathbf{E}_y^0\right)^2} = \frac{2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\cos\delta}{\left(\mathbf{E}_x^0\right)^2 - \left(\mathbf{E}_y^0\right)^2} \qquad \text{[VI-7c]}$$

$$\pm\frac{2\,a\,b}{a^2 + b^2} = \frac{2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\sin\left(\delta_y - \delta_x\right)}{\left(\mathbf{E}_x^0\right)^2 + \left(\mathbf{E}_y^0\right)^2} = \frac{2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\sin\delta}{\left(\mathbf{E}_x^0\right)^2 + \left(\mathbf{E}_y^0\right)^2} \qquad \text{[VI-7d]}$$

These relations are simplified if we introduce the two auxiliary angles

$$\tan\alpha = \mathbf{E}_y^0 / \mathbf{E}_x^0 , \qquad \text{[VI-8a]}$$

and

$$\tan\theta = \pm\, b/a , \qquad \text{[VI-8b]}$$

(Note: $\frac{b-a}{b}$ specifies the, so called, *ellipticity* of the vibrational ellipse.). In terms of these auxiliary, we may then write

$$\left(\mathbf{E}_x^0\right)^2 + \left(\mathbf{E}_y^0\right)^2 = a^2 + b^2 \qquad \text{[VI-9a]}$$

$$\tan 2\varphi = \frac{2\tan\alpha}{1-\tan^2\alpha}\cos\delta = \left(\tan 2\alpha\right)\cos\delta \qquad \text{[VI-9b]}$$

$$\sin 2\theta = \left(\sin 2\alpha\right)\sin\delta \qquad \text{[VI-9c]}$$

Probably, the most powerful representation of polarization is found in the famous Stokes vector or parameters -- *viz.*[24]

$$I = \left(\mathbf{E}_x^0\right)^2 + \left(\mathbf{E}_y^0\right)^2 \qquad \text{[VI-10a]}$$

$$M = \left(\mathbf{E}_x^0\right)^2 - \left(\mathbf{E}_y^0\right)^2 \qquad \text{[VI-10b]}$$

$$C = 2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\cos\left(\delta_y - \delta_x\right) = 2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\cos\delta \qquad \text{[VI-10c]}$$

$$S = 2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\sin\left(\delta_y - \delta_x\right) = 2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\sin\delta \qquad \text{[VI-10d]}$$

so that a polarized field may be represented by the vector

---

[24] G. G. Stokes, "On the composition and resolution of streams of polarized light from different sources," *Trans. Cambridge Phil. Soc. 9*, 399 (1852).

$$\begin{bmatrix} I \\ M \\ C \\ S \end{bmatrix} \qquad \text{[VI-11a]}$$

or its transpose $\{I \quad M \quad C \quad S\}$ [VI-11b]

These vector components give a complete geometric description of the vibrational ellipse -- *viz*.

I	Size
$C/M = \tan 2\varphi$	Azimuth
$\lvert S\rvert/I = \sin 2\theta = \dfrac{2b/a}{1+(b/a)^2}$	Shape
Sign of S	Handedness

As can see from Equations [VI-10], for a completely polarized field only three of these parameters or vector components are independent, since

$$I^2 = M^2 + C^2 + S^2 \qquad \text{[VI-9]}$$

and $\{M, C, S\}$ can be interpreted as the Cartesian coordinates of a sphere of radius I -- the Poincaré sphere.[25] Thus, from Equations [VI-8] and [VI-10] we obtain the coordinates of the Poincaré sphere or representation as

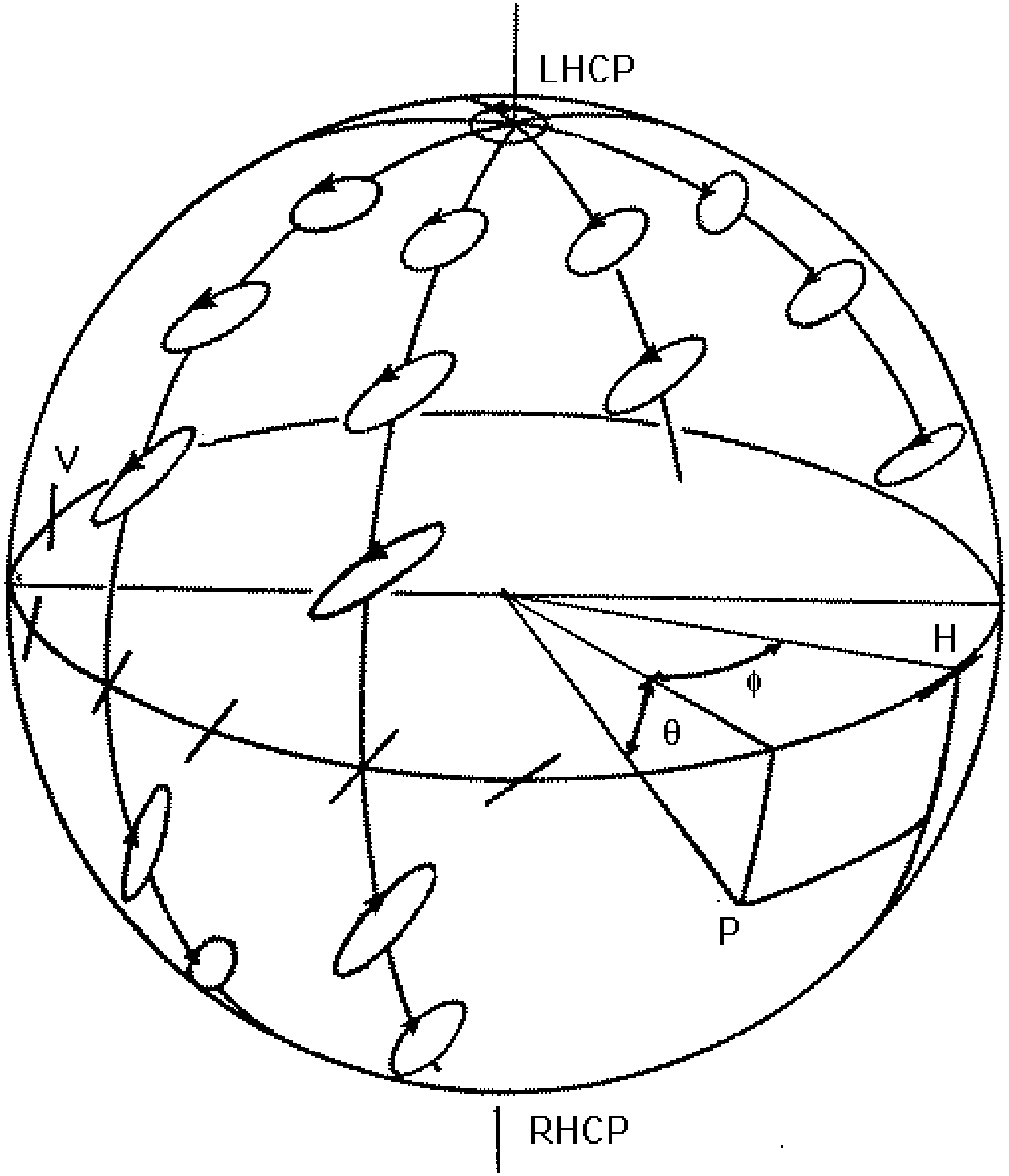
[25] H. Poincaré, *Théorie Mathématique de la Lumière*, Vol. 2, (1892) Chap. 12.

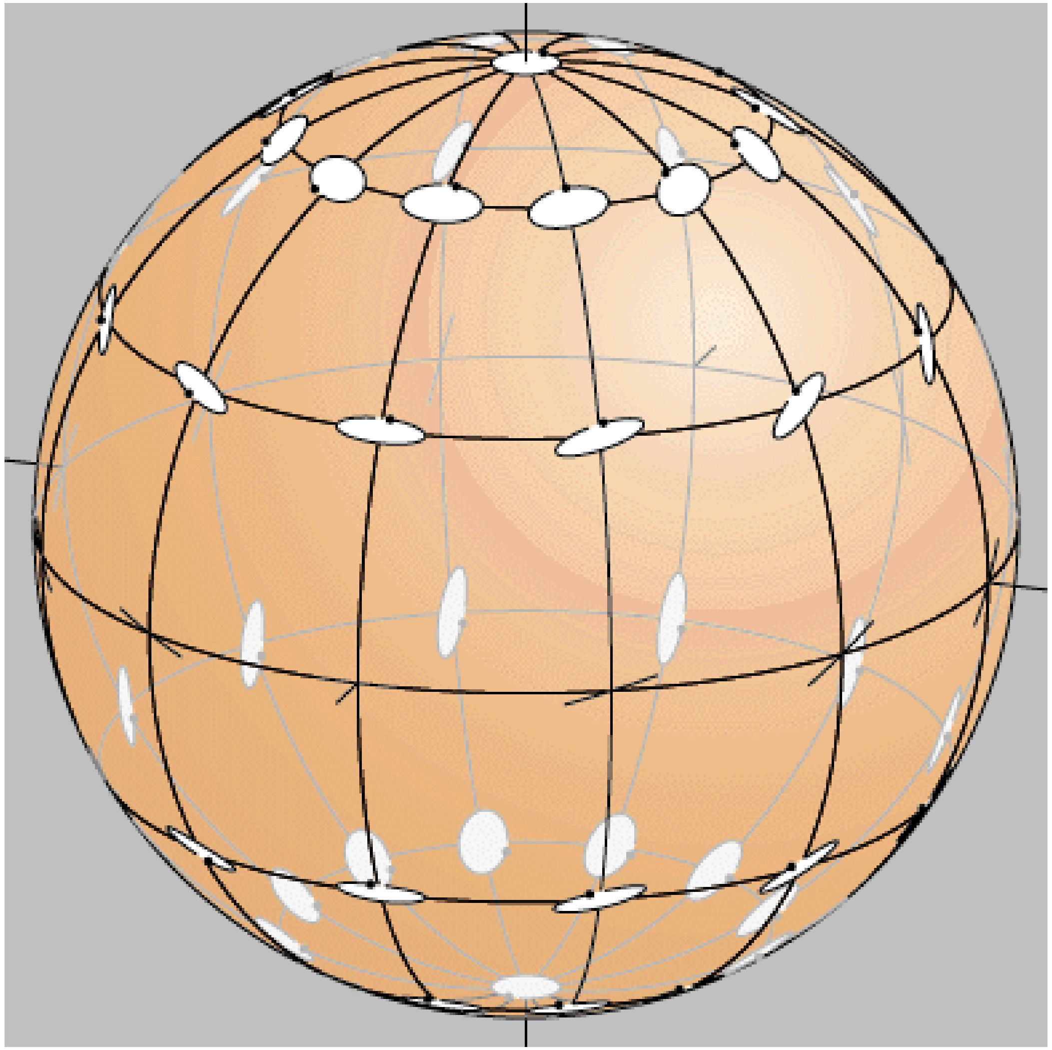
$$I^2 = M^2 + C^2 + S^2 \qquad \text{[VI-10a]}$$

$$M = I\cos 2\theta \ \cos 2\varphi \qquad \text{[VI-10b]}$$

$$C = I\cos 2\theta \sin 2\varphi \qquad \text{[VI-10b]}$$

$$S = I\sin 2\theta \qquad \text{[VI-10b]}$$


LHCP
V
H
φ
θ
P
RHCP



From: http://www.emeraldgraphics.com/Poincare.html